O P I B E

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A Various proposals have suggested that an adequate explanatory theory should reduce the number or the cardinality of the set of logically independent claims that need be accepted in order to entail a body of data. A (and perhaps the

"untransparent necessitations." It can be objected that theories have no such implications, that empirical results are only obtained by adding auxiliary hypotheses to them. This is mainly a matter of what is called a "theory." Celestial dynamics, for example, seems to illustrate Kneale's conception. The combination of Newtonian dynamics, the gravitational force law, and the claim that all forces between celestial bodies, considered as point particles, are gravitational, forms a finitely axiomatizable theory (or at least a finitely axiomatizable addition to a mathematical background), but has an infinity of consequences about how various numbers of bodies will move under sundry initial conditions. It is beside the point that this particular theory is false, and that we do not know how to calculate most of its implications analytically. It can be objected that some theories that are not finitely axiomatizable are nonetheless surveyable because an infinite collection of axioms are instances of a single axiom schema, for example the induction schema in Peano's postulates, or the continuity schema in first order Euclidean and Hyperbolic geometries. But Kneale was interested in reducing the cardinality of the set of assumptions, not only in surveyability or transparency. Finally, it can be objected that "observable" is notoriously vague. But in many cases in science a theory addresses a set of quantities with established criteria for their measurement, and for Kneale's proposal it is inessential whether they , equivalent to a single sentence, finer numbering distinctions than Kneale's would seem to require some counting principle for the number of putative laws in a sentence, or at least some ordering of explanations by the content they postulate. How should we count the *number* of "untransparent necessitations" in a finitely axiomatizable theory? Citing the passage above from Kneale, Friedman [5] attempted a counting principle for sets of sentences in terms of "independent acceptability." His proposal had various technical difficulties that seem to have proved insuperable [10].

If Kneale's criterion is too weak to connect explanation with truth, it seems too strong to allow explanations to be more probable than all that they would explain. Van Fraassen [20] has noted that it is (probabilistically) incoherent to assign a higher probability to a theory than to the collection of testable consequences of the theory. The theory, after all, entails all sentences in the collection, and if probability respects entailment then the theory can be no more probable than what it entails. So a theory meeting Kneale's criterion can never be more probable than its set of observational consequences. So if the probability ordering of sets of propositions were the proper and entire rational basis for a preference about which sets of propositions to believe or to accept, there could be no probabilistic grounds for preferring a Kneale theory to its set of observational consequences.

The obvious and decisive response to this argument is that the principle that we should accept or believe only the maximally probable set of propositions, regardless of informativeness or other virtues, would lead to accepting only logical truths.

A preference for more probable propositions or sets of propositions can be combined with other criteria, for example by first partially ordering propositions or sets of propositions by those criteria, and then refining that ordering by probability. On grounds other than probability, one can prefer theories meeting Kneale's criterion to their respective sets of testable consequences, but among theories meeting Kneale's criterion, probabilities can respect the partial ordering of purely deductive entailment. If theory T meeting Kneale's criterion deductively entails theory Q also satisfying that criterion, then Q must be at least as probable as T. Thus, for any collection of observationally equivalent theories that meet Kneale's criterion, a logically weakest such theory—one that is logically entailed by every Kneale theory that has the same observational consequences—would be maximally probable. So we have a proposal for a constraint on inference to the best explanation. Given any finite body58-8(rai(for)-v0vide)-8(aceabi)-10(lity)58(any)]TJT*(colle)5(vde)-8(a99(any)]TJT*(2l1(

What is the connection with truth? There is no guarantee that true theories meet Kneale's. criterion, and so no guarantee that we could use observational data to find the truth even in the limit as the number of logically independent observational claims increases without bound.[3] But there appears to be a proper subclass of theories one of which must be true if any theory meeting Kneale's criterion is true: the class of logically (i.e., deductively) weakest theories meeting Kneale's criterion. Since some logically weakest theory meeting Kneale's criterion would be true if any theory meeting the criterion and having the same testable consequences were true, we have a connection with truth—not all that we would like, but some—and a kind of pragmatic vindication. Of course, a preference for logically weakest theories does not solve the problem of induction from finite evidence, since finite evidence might be entailed by many, mutually inconsistent, logically weakest theories meeting Kneale's criterion. Further, for various not finitely axiomatizable classes of observation statements, it is at least conceivable that thebe dhebe.8625 .86-302cb-1.2ssersfy

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binary predicate [16, 19]. We may assume that all of the theories could in principle be formulated to extend the language of the testable sentences by the addition of claims formulated in terms of such a single extra predicate. But that does not show that best explanations exist.

The consequences of any best explanation, if it exists, will be a recursively enumerable collection of sentences. Consider any definite infinite, not finitely axiomatizable collection of potential data, and extensions of that collection by a finitely axiomatizable theory in extra predicates. The extension must be conservative, that is, it must entail, in the language of the data, all and only the sentences in the infinite collection and their consequences. Say a vocabulary for a first order language is finite if the set of predicate symbols, function symbols and constant symbols is finite. The questions are then:

1. *Under hat conditions on a recursi el enumerable but not finitel a iomati able set O of first order sentences in finite ocabular Lo does there e ist a finitel a iomati able conser ati e e tension T of O in e tra predicates?*
2. *Gi en a recursi el enumerable set O of first-order sentences in finite ocabular Lo ha ing a finitel a iomati able e tension T in language Lt containing e tra predicates not in Lo, such that T is a conser ati e e tension (o er Lo) of O,* under what conditions on *O does there e ist a logicall eakest such theor in Lt?*

The answer to question 1 is known for all but a special case. The next section reviews that answer and the structure of the theory construction in the proof, which is relevant to question 2. The final section answers question 2 as completely as the available answers to question 1.

2 T E T M K ' C

Kleene [11] showed that any first order theory with only infinite models has a finitely axiomatizable conservative extension in extra predicates, and Craig and Vaught [1] strengthened this result to the following: any first order recursively enumerable set of sentences with at most a finite number of non-isomorphic finite models has such an extension. The proof contains a recipe for constructing such theories which will be useful in the next section.

Assume an expanded language L of a first order language Lo and a consistent, recursively enumerable set O of sentences in Lo. $Q^{(N)}$ is a finitely axiomatizable fragment of number theory capable of representing all recursive functions [19]. Let $\Delta_1 \ldots \Delta_n \ldots$ be a recursive sequence of terms in $Q^{(N)}$. Since the Godel codings V, F, respectively, of the vocabulary and well-formed formulas of Lo are recursive, and, by Craig's Theorem [2], the set of Godel codings of the axioms of O are recursive, there are formulas Θ_1, Θ_2. Θ_3. Θ_4, in $Q^{(N)}$ that represent those classes:

QðNÞ|□ 1ðΔmÞ if m 2 F; QðNÞ|□ □ 1ðΔmÞ otherwise
QðNÞ|□ 2ðΔmÞ if m 2 V; QðNÞ|□ □ 2ðΔmÞ otherwise
QðNÞ|□ 3ðΔmÞ if m 2 Ax; ; QðNÞ|□ □ 3ðΔmÞ otherwise
QðNÞ|□ 4ðΔm, Δn, ΔpÞ if Cðm, nÞ ¼ p; QðNÞ|□ □ 4ðΔm, Δn, , ΔpÞ otherwise,

where "Cn" is the concatenation operation

The Craig-Vaught theory uses extra constant symbols, Po…Pp-1, Fm, Vb, Ax, As, and E,, the extra function symbols Cn, As, St, Vl, and the symbols of $Q^{(N)}$ (N, 0, +,),. The intended meaning of As(x) is "x is an assignment"; the meaning of Vl(x,u) = z is "the value of x at variable u is z"; and the meaning E(x, x', u, z) is "assignments x and x' have equal values at all variables except at most the variable u and the value of x' at u is z". Ax is the recursive predicate that holds of all and only the Godel numbers of the axioms of O represented by a formula as in Θ_3 above. The axioms of the Craig-Vaught theory are the universal closures of the following (which I quote, p. 296):

theory. Then, any sentence S having only infinite models must entail W. And conversely, if S entails W, then it must have only infinite models, because W has only infinite models. So the set of sentences with only infinite models is exactly the set of sentences entailing W. But the set of sentences entailing W is recursively enumerable (by enumerating the proofs starting with S, we can find that it entails W, if in fact it does), and by a theorem of Vaught's [21], the set of sentences with only infinite models is not recursively enumerable.

The negative result extends to all of the theories meeting the Craig-Vaught condition, although the argument is different. Starting with any recursively axiomatizable theory, O, Axiom 5 of the Craig Vaught theory for O then says that entailed by each theory in the sequence, and hence would also be in $\bigcap_n \mathrm{Con}(CVO_n)$. Hence $\bigcap_n \mathrm{Con}(CVO_n)$ would be finitely axiomatizable. By contradiction, there is no logically weakest Kneale theory for O.